ON THE MCKAY CORRESPONDENCES FOR THE HILBERT SCHEME OF POINTS ON THE AFFINE PLANE

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Abstract. The quotient of a finite-dimensional vector space by the action of a finite subgroup of automorphisms is usually a singular variety. Under appropriate assumptions, the McKay correspondence relates the geometry of nice resolutions of singularities and the representations of the group. For the Hilbert scheme of points on the affine plane, we study how different correspondences (McKay, dual McKay and multiplicative McKay) are related to each other.

1. Introduction

Let V be a finite-dimensional complex vector space and $G \subset SL(V)$ a finite subgroup of automorphisms. The quotient $V/G = \operatorname{Spec} \mathcal{O}(V)^G$ is usually a singular variety. *McKay correspondences* aim to relate the geometry of nice resolutions of singularities $Y \to V/G$ to the group G. Such correspondences were first constructed by McKay [37] and Gonzales-Sprinberg-Verdier [17] in dimension 2 and then generalized in dimension 3 by several authors (for a survey of the subject, see Reid [43] and references therein). In this paper, we are interested in generalizations to higher dimensions dues to Bridgeland-King-Reid [7], Kaledin [29, 30, 31], Ginzburg-Kaledin [16] and Bezrukavnikov-Kaledin [2].

We suppose that the vector space V is equipped with a symplectic form and that the group G preserves the symplectic structure. Suppose given a *crepant* resolution of singularities $Y \to V/G$ ($K_Y \cong \mathcal{O}_Y$). We study three constructions of McKay correspondences:

McKay correspondence. The derived categories of coherent sheaves $D^b(\mathrm{Coh}(Y))$ and $D^b(\mathrm{Coh}_G(V))$ are equivalent (Bezrukavnikov-Kaledin [2]). In some special cases, the *Hilbert scheme of regular G-orbits* $Y := G\text{-}\operatorname{Hilb} V$ provides such a resolution and the equivalence of categories is constructed as a Fourier-Mukai functor (Bridgeland-King-Reid [7]). This induces an isomorphism of Grothendieck groups $K(Y) \cong K_G(V)$.

Dual McKay correspondence. Recall Kaledin's results ([31]). The vector space V is naturally stratified by the subspaces of invariant vectors for different subgroups of G, inducing a stratification of V/G. The resolution $Y \to V/G$ is *semi-small* with respect to this stratification. The maximal strata are indexed by conjugacy classes in G and form a basis of the Borel-Moore homology with complex coefficients $H_*^{\mathrm{BM}}(Y)$. Denoting the space of $\mathbb{C}$-valued functions on G invariant by conjugation by $\mathcal{C}(G)$, we get a linear isomorphism $H_*^{\mathrm{BM}}(Y) \cong \mathcal{C}(G)$.

Multiplicative McKay correspondence. The increasing filtration of the group algebra $F^d\mathbb{C}[G] := \mathbb{C}\{g \in G \mid \operatorname{rg}(id_V - g) \leq d\}$ is compatible with the ring structure.

1991 *Mathematics Subject Classification.* Primary 14C05; Secondary 05E05,20B30,55N91.
Key words and phrases. Hilbert scheme, McKay correspondence, symmetric functions, equivariant cohomology, Macdonald polynomials.

By graduation and restriction to the center ZG, we get a graded commutative algebra $\operatorname{gr}^F ZG$. There is a natural isomorphism of graded algebras $H^*(Y) \cong \operatorname{gr}^F ZG$ (Ginzburg-Kaledin [16]).

A natural question occurs at this point: how are these three correspondences related to each other? In order to compare them, one is lead to the following two problems, formulated by Ginzburg-Kaledin [16]:

Poincaré duality problem. ([16, Problem 1.4]) Compute the Poincaré duality isomorphism:

$$\operatorname{gr}^F ZG \longrightarrow H^*(Y) \xrightarrow{D} H_*^{\mathrm{BM}}(Y) \longrightarrow \mathcal{C}(G).$$

Chern character problem. ([16, Problem 1.5]) Compute the Chern character isomorphism:

$$ZG \cong R(G) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong K_G(V) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \longrightarrow K(Y) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \xrightarrow{ch} H^*(Y) \longrightarrow \operatorname{gr}^F ZG.$$

We study these questions in the particular case $V = \mathbb{C}^n \otimes \mathbb{C}^2$ with the permutation action of the symmetric group $G = S_n$ and the canonical symplectic structure. The Hilbert scheme $Y = \operatorname{Hilb}^n(\mathbb{C}^2)$ provides a natural symplectic resolution of singularities, isomorphic to the Hilbert scheme of regular orbits $S_n\text{-}\operatorname{Hilb}\mathbb{C}^{2n}$. Here, the *McKay correspondence* is realized by the Bridgeland-King-Reid theorem [7] and computed with Haiman's results ([21, 23]); the multiplicative McKay correspondence is constructed by Nakajima's operators ([40]) and the theorems of Lehn-Sorger [32] and Vasserot [44] and finally the *dual McKay correspondence* is given by natural subvarieties inducing a basis of the Borel-Moore homology. In this context, we can replace the spaces $R(S_n)$, $\mathcal{C}(S_n)$ and ZS_n by the space of *symmetric functions* Λ^n and work with rational coefficients. We use the natural basis of *Newton functions* p_λ and *Schur functions* s_λ indexed by *partitions* λ of n. We define a graduation by $\deg p_\lambda := n - l(\lambda)$ where $l(\lambda)$ is the *length* of the partition and consider the decreasing filtration $F_d\Lambda^n := \mathbb{Q}\{p_\lambda \mid \deg p_\lambda \geq d\}$. For any symmetric function $f \in \Lambda^n$ we denote the homogeneous component of degree k of f by $[f]_k$.

In this setup, the *Poincaré duality problem* consists in the computation of the map:

$$\gamma : \Lambda^n \longrightarrow H^*(\operatorname{Hilb}^n(\mathbb{C}^2), \mathbb{Q}) \xrightarrow{D} H_*^{\mathrm{BM}}(\operatorname{Hilb}^n(\mathbb{C}^2), \mathbb{Q}) \longrightarrow \Lambda^n$$

and the *Chern character problem* consists in the computation of the map:

$$\Gamma : \Lambda^n \longrightarrow K(\operatorname{Hilb}^n(\mathbb{C}^2)) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \xrightarrow{ch} H^*(\operatorname{Hilb}^n(\mathbb{C}^2), \mathbb{Q}) \longrightarrow \Lambda^n.$$

Denote the *age* (or *shifting degree*) of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n by $\operatorname{age}(\lambda) := n - l(\lambda)$ and define his *complexity degree* as $\langle\lambda\rangle := \prod_{i \geq 1} \lambda_i$. These numbers can be given a more general definition for any group acting on a vector space: the *age* is computed with a diagonalization of the action whereas the *complexity degree* is defined by the Frobenius decomposition of the action.

We prove the following formulas (§5.1, §5.2):

Theorem 1.1. *For any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n,*

$$\gamma(p_\lambda) = \frac{(-1)^{\operatorname{age}(\lambda)}}{\langle\lambda\rangle} p_\lambda,$$

$$\Gamma(p_\lambda) = \frac{(-1)^{\operatorname{age}(\lambda)}}{\langle\lambda\rangle} p_\lambda + \sum_{\substack{\mu \vdash n \\ l(\mu) < l(\lambda)}} g_{\lambda,\mu} p_\mu,$$

for some coefficients $g_{\lambda,\mu}$.

Moreover we show the following theorem (§5.3):

Theorem 1.2. *The McKay correspondence is compatible with the decreasing topological filtration of* $K(\mathrm{Hilb}^n(\mathbb{C}^2))$ *and the decreasing filtration of* Λ^n*.*

Hence we can graduate the McKay correspondence and get a *graded McKay correspondence*. Our formulas then show that surprisingly, up to natural identifications, the *graded McKay correspondence*, the *multiplicative McKay correspondence* and the *dual McKay correspondence* are the same.

The main tool in our computations consists in computing combinatorial formulas for the Chern classes of vector bundles on $\mathrm{Hilb}^n(\mathbb{C}^2)$ linearizing the natural action of the torus $\mathbb{C}^*$ (§4.2):

Theorem 1.3. *Let* F *be a* $\mathbb{C}^*$*-linearized vector bundle of rank* r *on* $\mathrm{Hilb}^n(\mathbb{C}^2)$ *and* $f_1^\lambda, \ldots, f_r^\lambda$ *the weights of the action on the fibre at each fixed point. Then the Chern classes of* F *written in* Λ^n *are*

$$c_k(F) = \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \sigma_k(f_1^\lambda, \ldots, f_r^\lambda)[s_\lambda]_k,$$

where the $\sigma_k(-)$ *are the elementary symmetric functions.*

The Chern characters of F *are*

$$ch_k(F) = \frac{1}{k!} \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \sum_{i=1}^{r} \left(f_i^\lambda\right)^k [s_\lambda]_k.$$

2. Symmetric functions

2.1. **The ring of symmetric functions.** ([35, 36])

Take independent indeterminates $x_1, \ldots, x_r$. Let the symmetric group S_r act by permutation on $\mathbb{Q}[x_1, \ldots, x_r]$ and denote the invariant ring by $\Lambda_r := \mathbb{Q}[x_1, \ldots, x_r]^{S_r}$. This ring is naturally graded by degree and we denote the vector subspace generated by degree n homogeneous symmetric polynomials by Λ_r^n. By adjoining other indeterminates, we may construct the projective limit $\Lambda^n := \varprojlim \Lambda_r^n$. Then the *ring of symmetric functions* is defined by $\Lambda := \bigoplus_{n \geq 0} \Lambda^n$.

A *partition* of an integer n is a decreasing sequence of non-negative integers $\lambda := (\lambda_1, \ldots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = n$ (we write $\lambda \vdash n$). The λ_i are the *parts* of the partition. If necessary, we extend a partition with zero parts. The number $l(\lambda)$ of non-zero parts is the *length* of the partition and the sum $|\lambda|$ of the parts is the *weight*. If a partition λ has α_1 parts equal to 1, α_2 parts equal to 2, … we shall also denote it by $\lambda := (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$.

The *Young diagram* of a partition λ is defined by

$$D(\lambda) := \{(i,j) \in \mathbb{N} \times \mathbb{N} \,|\, j < \lambda_{i+1}\}.$$

In the representation of such a diagram, we follow a matrix convention:

$$\begin{array}{|c|c|c|c|} \hline & x & h & h \\ \hline & h & & \multicolumn{1}{c}{} \\ \cline{1-3} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{cc} \lambda = (4,3,1) & x = (0,1) \\ |\lambda| = 8 & h(x) = 4 \\ l(\lambda) = 3 & \end{array}$$

For each cell $x \in D(\lambda)$, the *hook length* $h(x)$ at x is the number of cells on the right and below x. We shall also make use of the number $n(\lambda) := \sum_{i \geq 1} (i-1)\lambda_i$.

Set $p_0 = 1$ and define for $k \geq 1$ the *power sum* $p_k := \sum_{i \geq 1} x_i^k$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, the *Newton function* is the product $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_k} \in \Lambda^{|\lambda|}$. The

Newton functions form a basis of Λ and $\Lambda \cong \mathbb{Q}[p_1, p_2, \ldots]$. Another natural basis of Λ indexed by partitions is given by the *Schur functions* s_λ.

For a partition $\lambda = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$, set $z_\lambda := \prod_{r\geq 1} \alpha_r! r^{\alpha_r}$ and define a scalar product on Λ by $\langle p_\lambda, p_\mu\rangle = \delta_{\lambda,\mu} z_\lambda$ where $\delta_{\lambda,\mu}$ is the Kronecker symbol. Then the basis of Schur functions is orthonormal.

Let $\mathcal{C}(S_n)$ be the $\mathbb{Q}$-vector space of *class functions* on S_n. Since conjugacy classes in S_n are indexed by partitions, the functions χ_λ taking the value 1 on the conjugacy class λ and 0 else form a basis of $\mathcal{C}(S_n)$. Let $R(S_n)$ be the $\mathbb{Q}$-vector space of representations of S_n. By associating to each representation of S_n his *character*, we get an isomorphism $\chi : R(S_n) \to \mathcal{C}(S_n)$. The *Frobenius morphism* is the isomorphism $\Phi : \mathcal{C}(S_n) \to \Lambda^n$ characterized by $\Phi(\chi_\lambda) = z_\lambda^{-1} p_\lambda$. Let χ^λ be the class function such that $\Phi(\chi^\lambda) = s_\lambda$ and χ^λ_μ the value of χ^λ at the conjugacy class μ. The representations V^λ of character χ^λ are the irreducible representations of S_n and we have the following base-change formulas:

$$p_\mu = \sum_{\lambda\vdash n} \chi^\lambda_\mu s_\lambda \text{ (Frobenius formula)},$$
$$s_\lambda = \sum_{\mu\vdash n} z_\mu^{-1} \chi^\lambda_\mu p_\mu \text{ (inverse Frobenius formula)}.$$

2.2. **Phethystic substitutions.** ([20, 35])

The identification $\Lambda = \mathbb{Q}[p_1, p_2, \ldots]$ allows to specialize the p_k's to elements of any $\mathbb{Q}$-algebra: the specialization extends uniquely to an algebra homomorphism on Λ. For a formal Laurent series E in indeterminates $t_1, t_2, \ldots$ we define $p_k[E]$ to be the result of replacing each indeterminate t_i by t_i^k. Extending the specialization to any symmetric function $f \in \Lambda$, we obtain the *plethystic substitution* of E in f, denoted by $f[E]$. Our convention is that in a plethystic substitution, X stands for the sum of the original indeterminates $x_1 + x_2 + \cdots$, so that $p_k[X] = p_k$.

2.3. **Macdonald polynomials.** ([20, 34])

We introduce indeterminates q, t and consider the ring $\Lambda_{\mathbb{Q}(q,t)} := \Lambda \underset{\mathbb{Q}}{\otimes} \mathbb{Q}(q, t)$. The scalar product and the plethystic substitutions naturally extend to this situation. For a partition μ, we define $B_\mu(q, t) := \sum_{(i,j)\in D(\mu)} t^i q^j$. Set

$$\Omega := \exp\left(\sum_{k\geq 1} \frac{p_k}{k}\right)$$

and define a linear operator $\Delta : \Lambda_{\mathbb{Q}(q,t)} \to \Lambda_{\mathbb{Q}(q,t)}$ by

$$\Delta f = f\left[X + \frac{(1-t)(1-q)}{z}\right] \Omega[-zX]\bigg|_{z^0}.$$

The *modified Macdonald polynomial* $\widetilde{H}_\mu$ is the eigenvector of Δ corresponding to the eigenvalue $1 - (1 - q)(1 - t)B_\mu(q, t)$. These polynomials form a basis of $\Lambda_{\mathbb{Q}(q,t)}$ and decompose in the basis of Schur functions as:

$$\widetilde{H}_\mu = \sum_{\lambda\vdash n} \widetilde{K}_{\lambda,\mu} s_\lambda,$$

where the $\widetilde{K}_{\lambda,\mu} \in \mathbb{N}[q, t]$ are called *q, t-Kostka polynomials*. We shall make use of the following specialization at $t = 1/q$ ([24, Proposition 3.5.10]):

$$\widetilde{H}_\mu(q, q^{-1}) = q^{-n(\mu)} \prod_{x\in D(\mu)} \left(1 - q^{h(x)}\right) s_\mu\left[\frac{X}{1-q}\right]. \tag{1}$$

3. Hilbert schemes in the affine plane

3.1. **Hilbert scheme of points.** ([6, 13, 14, 15, 18, 32, 39, 40, 44])

The *Hilbert scheme of n points in the affine plane* $\mathrm{Hilb}^n(\mathbb{C}^2)$ is the smooth quasi-projective manifold of complex dimension $2n$ parameterizing length n finite subschemes in the plane $\mathbb{C}^2$. The first projection $B_n := pr_{1*}\mathcal{O}_{\Xi_n}$ of the universal family $\Xi_n \subset \mathrm{Hilb}^n(\mathbb{C}^2)\times\mathbb{C}^2$ is the rank n *usual tautological bundle* on $\mathrm{Hilb}^n(\mathbb{C}^2)$.

The manifold $\mathrm{Hilb}^n(\mathbb{C}^2)$ has no odd singular cohomology; his even cohomology has no torsion and is generated by algebraic cycles. We denote the cohomology ring by $H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$, the Borel-Moore homology by $H_*^{\mathrm{BM}}(\mathrm{Hilb}^n(\mathbb{C}^2))$ and the Grothendieck group of algebraic vector bundles (or equivalently of coherent sheaves) by $K(\mathrm{Hilb}^n(\mathbb{C}^2))$, all with rational coefficients. Denote the Chern character by $ch : K(\mathrm{Hilb}^n(\mathbb{C}^2)) \xrightarrow{\sim} H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$ and the Poincaré duality by $D : H^*(\mathrm{Hilb}^n(\mathbb{C}^2)) \xrightarrow{\sim} H_*^{\mathrm{BM}}(\mathrm{Hilb}^n(\mathbb{C}^2))$.

Denote the Mumford quotient parameterizing length n effective zero-cycles in $\mathbb{C}^2$ by $S^n\mathbb{C}^2 := \mathbb{C}^{2n}/S_n$. The *Hilbert-Chow* morphism $\rho : \mathrm{Hilb}^n(\mathbb{C}^2) \to S^n\mathbb{C}^2$ is a symplectic resolution of singularities, semi-small with respect to the natural stratification $S_\lambda\mathbb{C}^2 := \left\{\sum_{i=1}^{k}\lambda_i x_i \,|\, x_i \neq x_j \text{ for } i\neq j\right\}$ for partitions $\lambda = (\lambda_1,\dots,\lambda_k)$ of n. Each subvariety $X_\lambda := \rho^{-1}S_\lambda\mathbb{C}^2$ is irreducible and locally closed of dimension $n + l(\lambda)$.

There is a natural isomorphism

$$\Psi : \Lambda^n \longrightarrow H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$$

constructed by use of geometric operators acting on the total sum of cohomology of Hilbert schemes. For $i \geq 1$ denote by $X_{n,i} \subset \mathrm{Hilb}^n(\mathbb{C}^2)\times\mathrm{Hilb}^{n+i}(\mathbb{C}^2)$ the subvariety of nested subschemes $\xi \subset \xi'$ such that ξ and ξ' differ by a point of length i. Let π_n, π_{n+i} be the respective projections on $\mathrm{Hilb}^n(\mathbb{C}^2)$ and $\mathrm{Hilb}^{n+i}(\mathbb{C}^2)$ and define the operator

$$q_i : H^*(\mathrm{Hilb}^n(\mathbb{C}^2)) \longrightarrow H^{*+2i-2}(\mathrm{Hilb}^{n+i}(\mathbb{C}^2))$$

by[1] $q_i(\alpha) = \pi_{n+i!}(\pi_n^*(\alpha)\cup[X_{n,i}])$. Denote the unit by $|0\rangle \in H^0(\mathrm{Hilb}^0(\mathbb{C}^2))$. Nakajima [40] proves that the vectors

$$q_\lambda := q_{\lambda_1}\circ\cdots\circ q_{\lambda_k}|0\rangle \in H^{2n-2l(\lambda)}(\mathrm{Hilb}^n(\mathbb{C}^2))$$

where $\lambda = (\lambda_1,\dots,\lambda_k)$ runs over all partitions of n form a basis of $H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$. The isomorphism Ψ is defined by $\Psi(p_\lambda) = q_\lambda$. We shall make use of the following observation concerning the cohomology classes of the subvarieties X_λ:

$$\text{for } \lambda = (1^{\alpha_1}, 2^{\alpha_2},\dots),\quad \left[\overline{X_\lambda}\right] = \frac{1}{\prod_{i\geq1}\alpha_i!}q_\lambda. \tag{2}$$

We now describe the ring structure we have on $H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$ as explained in Lehn-Sorger [32] and Vasserot [44]. Introduce the following graduation of the group algebra $\mathbb{Q}[S_n]$. For a permutation π of cycle-type λ, set $\deg(\pi) := n - l(\lambda)$ and denote the vector subspace generated by degree d elements by $\mathbb{Q}[S_n](d)$. The natural ring structure on $\mathbb{Q}[S_n]$ is not compatible with the graduation but with the associated increasing filtration:

$$F^d\mathbb{Q}[S_n] := \bigoplus_{d'\leq d}\mathbb{Q}[S_n](d').$$

[1] For any continuous map $f : X \to Y$ between smooth oriented manifolds, we denote the push-forward map in cohomology induced from the homological push-forward by Poincaré duality by $f_! : H^*(X) \to H^*(Y)$.

We consider the associated graded ring $\operatorname{gr}^F \mathbb{Q}[S_n] := \bigoplus_{d=0}^{n-1} F^d\mathbb{Q}[S_n]/F^{d-1}\mathbb{Q}[S_n]$. The center ZS_n of $\mathbb{Q}[S_n]$ is generated by the homogeneous elements χ_λ so it inherits graduation, filtration and ring structure. Denote by $\operatorname{gr}^F \Lambda^n$ the corresponding ring via the Frobenius isomorphism. The space $\operatorname{gr}^F \Lambda^n$ is graded by the *cohomological degree* $\deg p_\lambda := n - l(\lambda)$ and the increasing filtration is denoted by

$$F^d\Lambda^n := \mathbb{Q}\left\{p_\lambda \mid \deg p_\lambda \leq d\right\}.$$

Then $\Psi : \operatorname{gr}^F \Lambda^n \longrightarrow H^{2*}(\operatorname{Hilb}^n(\mathbb{C}^2))$ is a isomorphism of graded algebras (Lehn-Sorger [32], Vasserot [44]).

3.2. **Hilbert scheme of regular orbits.** ([7, 21, 26, 27, 41])

The *Hilbert scheme of S_n-regular orbits* $S_n\text{-}\operatorname{Hilb}\mathbb{C}^{2n}$ is defined as the closure in the Hilbert scheme $\operatorname{Hilb}^{n!}(\mathbb{C}^{2n})$ of the open set of S_n-free orbits and is isomorphic to the Hilbert scheme $\operatorname{Hilb}^n(\mathbb{C}^2)$ (Haiman [21, Theorem 5.1]). Denote the universal family by $Z_n \subset S_n\text{-}\operatorname{Hilb}\mathbb{C}^{2n} \times \mathbb{C}^{2n}$ and set $P_n := p_*\mathcal{O}_{Z_n}$ considered as the rank $n!$ *unusual tautological bundle* on $\operatorname{Hilb}^n(\mathbb{C}^2)$. This bundle is equipped with a natural S_n-action inducing the regular representation on each fiber. Consider the diagram:

$$\begin{tikzcd} Z_n \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{C}^{2n} \arrow[d] \\ \operatorname{Hilb}^n(\mathbb{C}^2) \arrow[r, "\rho"] & S^n\mathbb{C}^2 \end{tikzcd}$$

Denote by $D^b(\operatorname{Hilb}^n(\mathbb{C}^2))$ the derived category of coherent sheaves and $D^b_{S_n}(\mathbb{C}^{2n})$ the derived category of coherent S_n-sheaves. In this situation, we can apply the Bridgeland-King-Reid theorem ([7]) and get an isomorphism of Grothendieck groups

$$\Upsilon := q_! \circ p^! : K(\operatorname{Hilb}^n(\mathbb{C}^2)) \to K_{S_n}(\mathbb{C}^{2n}).$$

Consider the following composition of vector space isomorphisms:

$$\Theta : \Lambda^n \xrightarrow{\Phi^{-1}} \mathcal{C}(S_n) \xrightarrow{\chi^{-1}} R(S_n) \xrightarrow{\tau^{-1}} K_{S_n}(\mathbb{C}^{2n}) \xrightarrow{\Upsilon^{-1}} K(\operatorname{Hilb}^n(\mathbb{C}^2))$$

where τ is the Thom isomorphism (here it is the restriction to a fibre, see [11, Theorem 5.4.17]). The S_n-action on P_n induces an isotypical decomposition

$$P_n = \bigoplus_{\mu\vdash n} \mathbf{V}^\mu \otimes P_\mu$$

where $\mathbf{V}^\mu$ is the trivial bundle with fibre V^μ on $\operatorname{Hilb}^n(\mathbb{C}^2)$ and $P_\mu := \operatorname{Hom}_{S_n}(\mathbf{V}^\mu, P_n)$. Then a easy computation similar to [27, Formula (5.3)] shows that:

Proposition 3.1. *For $\mu \vdash n$, $\Theta(s_\mu) = P^*_\mu$. In particular, the dual bundles P^*_μ form a basis of $K(\operatorname{Hilb}^n(\mathbb{C}^2))$.*

3.3. **Torus action on the Hilbert scheme of points.** ([14, 23])

The torus $T := \mathbb{C}^*$ acts on $\mathbb{C}[x, y]$ by $s.x = sx, s.y = s^{-1}y$ for $s \in T$. It induces a natural action on $\operatorname{Hilb}^n(\mathbb{C}^2)$ with finitely many fixed points ξ_λ parameterized by the partitions λ of n. The action extends to all natural objets at issue over $\mathbb{C}^2$.

Let F be a T-linearized vector bundle on $\operatorname{Hilb}^n(\mathbb{C}^2)$. Each fibre $F(\xi_\lambda)$ has a structure of representation of T and by identifying the representation ring of T with the ring of polynomials $R(T) \cong \mathbb{Z}[s, s^{-1}]$ we set

$$F(\xi_\lambda) := F_\lambda(s) := \sum_{i=1}^{r} s^{f_i^\lambda},$$

where $f_i^\lambda \in \mathbb{Z}$ are the *weights* of the action of T on $F(\xi_\lambda)$.

In particular, we have the following result:

Proposition 3.2 (Haiman)**.** ([23, Proposition 3.4]) *For* $\lambda, \mu \vdash n$,

$$P_\mu(\xi_\lambda) = \widetilde{K}_{\mu,\lambda}\Big|_{t=s,q=s^{-1}}.$$

4. Chern classes of linearized bundles

4.1. **Equivariant cohomology of the Hilbert scheme of points in the affine plane.** ([9, 38, 44])

Let $E_T \to B_T$ be the classifying bundle of T-vector bundles. For any algebraic variety X with an action of T, let $H^*_T(X)$ and $H^T_*(X)$ be the equivariant cohomology and the equivariant Borel-Moore homology with rational coefficients. By definition, $H^*_T(X) = H^*(X_T)$ where $X_T := (X \times E_T)/B_T$. The ring $H^*_T(pt)$ is isomorphic to $\mathbb{Q}[u]$ where u is an indeterminate of degree 2 and $H^*_T(X)$ is a graded $\mathbb{Q}[u]$-algebra. We denote the unit by $1_X \in H^0_T(X)$. If X is smooth of pure dimension d, the Poincaré duality $D : H^i_T(X) \to H^T_{d-i}(X)$ is an isomorphism and for a proper T-equivariant morphism $f : Y \to X$ we have a push-forward morphism $f_! : H^*_T(Y) \to H^*_T(X)$. In particular, any closed T-stable subvariety $Y \overset{j}{\hookrightarrow} X$ defines a cohomology class $[Y]_T := j_! 1_Y \in H^*_T(X)$. Any T-linearized vector bundle F on X has T-equivariant Chern classes $c^T_k(F)$ and T-equivariant Chern characters $ch^T_k(F)$ in $H^{2k}_T(X)$ such that if $j : X \hookrightarrow X_T$ is the inclusion of a fibre, we have $c_k(F) = j^* c^T_k(F)$ and $ch_k(F) = j^* ch^T_k(F)$. For any $\mathbb{Q}[u]$-module M we denote the localization of M at the ideal $\langle u-1\rangle$ by M'.

Let $\Sigma \subset \mathbb{C}^2$ be the vertical axis and $X^\Sigma_{n,i}$ the subvariety of $X_{n,i}$ whose points are nested subschemes $\xi \subset \xi'$ with extremal point on Σ. As in Vasserot [44], define for $i \geq 1$ the operator

$$q^T_i[\Sigma] : H^*_T(\mathrm{Hilb}^n(\mathbb{C}^2)) \longrightarrow H^{*+2i}_T(\mathrm{Hilb}^{n+i}(\mathbb{C}^2))$$

by $q^T_i[\Sigma](\alpha) = \pi_{n+i!}(\pi^*_n(\alpha) \cup [X^\Sigma_{n,i}]_T)$. Vasserot [44] proves that the vectors:

$$q^T_\lambda[\Sigma] := q^T_{\lambda_1}[\Sigma] \circ \cdots \circ q^T_{\lambda_k}[\Sigma] 1_{\mathrm{Hilb}^0(\mathbb{C}^2)} \in H^{2n}_T(\mathrm{Hilb}^n(\mathbb{C}^2))$$

for all partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n form a basis of $H^{2n}_T(\mathrm{Hilb}^n(\mathbb{C}^2))$ and constructs an isomorphism

$$\phi : H^{2n}_T(\mathrm{Hilb}^n(\mathbb{C}^2)) \longrightarrow \Lambda^n$$

by[2] $\phi(q^T_\lambda[\Sigma]) = p_\lambda$.

Define for $i \geq 1$ the operator

$$q^T_i : H^*_T(\mathrm{Hilb}^n(\mathbb{C}^2)) \longrightarrow H^{*+2i-2}_T(\mathrm{Hilb}^{n+i}(\mathbb{C}^2))$$

by $q^T_i(\alpha) = \pi_{n+i!}(\pi^*_n(\alpha) \cup [X_{n,i}]_T)$ and set

$$q^T_\lambda := q^T_{\lambda_1} \circ \cdots \circ q^T_{\lambda_k} 1_{\mathrm{Hilb}^0(\mathbb{C}^2)} \in H^{2n-2l(\lambda)}_T(\mathrm{Hilb}^n(\mathbb{C}^2)).$$

Since $[\Sigma]_T = u.[\mathbb{C}^2]_T$ we see that $q^T_\lambda[\Sigma] = u^{l(\lambda)} q^T_\lambda$. The inclusion of a fibre $j : \mathrm{Hilb}^n(\mathbb{C}^2) \hookrightarrow (\mathrm{Hilb}^n(\mathbb{C}^2))_T$ gives $j^* q^T_\lambda = q_\lambda$. We can apply the Leray-Hirsch theorem to the situation:

$$\mathrm{Hilb}^n(\mathbb{C}^2) \xrightarrow{j} (\mathrm{Hilb}^n(\mathbb{C}^2))_T \xrightarrow{p} B_T$$

and get an isomorphism of graded $H^*(B_T)$-moduls:

$$H^*_T(\mathrm{Hilb}^n(\mathbb{C}^2)) \cong H^*(B_T) \otimes H^*(\mathrm{Hilb}^n(\mathbb{C}^2)).$$

[2] There is an inaccuracy in [44] : a factor $z_{(i)}$ is missing in the formula (2) (see Nakajima [39, Lemma 9.4]).

Since B_T and $\mathrm{Hilb}^n(\mathbb{C}^2)$ have no odd cohomology, the theorem gives a basis in each cohomological degree:

$$\begin{array}{ccc} H_T^{2m}(\mathrm{Hilb}^n(\mathbb{C}^2)) & \cong & \bigoplus_{k=0}^{m} H^{2m-2k}(B_T)\otimes H^{2k}(\mathrm{Hilb}^n(\mathbb{C}^2)) \\ u^{m-k}q_\lambda^T & \leftrightarrow & u^{m-k}\otimes q_\lambda \end{array}$$

where λ is a partition of n such that $n-l(\lambda)=k$.

The multiplication by u sending $H_T^k(\mathrm{Hilb}^n(\mathbb{C}^2))$ to $H_T^{k+2}(\mathrm{Hilb}^n(\mathbb{C}^2))$ is always injective. Since $H^q(\mathrm{Hilb}^n(\mathbb{C}^2))=0$ for $q\geq 2n$, the vector space $H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))$ contains all the information about the equivariant cohomology and multiplication by u becomes an isomorphism after this degree:

$$H_T^0(\mathrm{Hilb}^n(\mathbb{C}^2))\xrightarrow{u\cdot}\cdots\xrightarrow{u\cdot}H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))\underset{\sim}{\xrightarrow{u\cdot}}H_T^{2n+2}(\mathrm{Hilb}^n(\mathbb{C}^2))\underset{\sim}{\xrightarrow{u\cdot}}\cdots.$$

The Leray-Hirsch decomposition makes $H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))$ a graded vector space: we denote by $\mathrm{gr}\,H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))$ the vector space with his graded structure. The vectors $q_\lambda^T[\Sigma]$ form a homogeneous basis with $\deg q_\lambda^T[\Sigma]=n-l(\lambda)$. By the choice of this basis, we have a canonical isomorphism

$$H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))\xrightarrow{can.}\mathrm{gr}\,H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))$$

cutting up a vector in homogeneous components: for $\alpha\in H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))$ we denote the component of degree k in α by $\mathrm{gr}_k\,\alpha$. We have an isomorphism of graded vector spaces

$$J:\mathrm{gr}\,H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))\longrightarrow H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$$

defined by $J(q_\lambda^T[\Sigma])=q_\lambda$. The morphism $\phi: H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))\to\Lambda^n$ also induces a isomorphism of graded vector spaces $\mathrm{gr}\,\phi:\mathrm{gr}\,H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))\to\mathrm{gr}\,\Lambda^n$ and the following diagram is commutative:

$$\begin{tikzcd} \Lambda^n \arrow[r, "can."] & \mathrm{gr}\,\Lambda^n \arrow[dr, "\Psi"] & \\ H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2)) \arrow[u, "\phi"] \arrow[r, "can."] & \mathrm{gr}\,H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2)) \arrow[u, "\mathrm{gr}\,\phi"] \arrow[r, "J"] & H^*(\mathrm{Hilb}^n(\mathbb{C}^2)) \end{tikzcd}$$

Define $[\lambda]\in H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))$ by $u^n\cdot[\lambda]=(-1)^n h(\lambda)^{-1}[\xi_\lambda]_T$ where $h(\lambda)$ is the product of the hook lengths in the Young diagram of λ. We have:

Proposition 4.1 (Vasserot)**.** ([44]) *For $\lambda\vdash n$, $\phi([\lambda])=s_\lambda$.*

4.2. **Chern classes of linearized bundles.**

In the study of the Chern classes of natural vector bundles on $\mathrm{Hilb}^n(\mathbb{C}^2)$, we prove the following formulas:

Theorem 4.2. *Let F be a T-linearized vector bundle of rank r on $\mathrm{Hilb}^n(\mathbb{C}^2)$ and $f_1^\lambda,\ldots,f_r^\lambda$ the weights of the action on the fibre at each fixed point. Then the Chern classes of F written in Λ^n via Ψ are*

$$c_k(F)=\sum_{\lambda\vdash n}\frac{1}{h(\lambda)}\sigma_k(f_1^\lambda,\ldots,f_r^\lambda)\sum_{\substack{\mu\vdash n\\ l(\mu)=n-k}}z_\mu^{-1}\chi_\mu^\lambda p_\mu,$$

where the $\sigma_k(-)$ are the elementary symmetric functions.

The Chern characters of F are

$$ch_k(F)=\frac{1}{k!}\sum_{\lambda\vdash n}\frac{1}{h(\lambda)}\sum_{i=1}^{r}\left(f_i^\lambda\right)^k\sum_{\substack{\mu\vdash n\\ l(\mu)=n-k}}z_\mu^{-1}\chi_\mu^\lambda p_\mu.$$

Proof. The inclusion of a fixed point is denoted by $i_\lambda : \xi_\lambda \hookrightarrow \mathrm{Hilb}^n(\mathbb{C}^2)$ and we set $[\xi_\lambda]_T := i_{\lambda !} 1_{\xi_\lambda} \in H_T^{4n}(\mathrm{Hilb}^n(\mathbb{C}^2))$. The inclusion of the fixed points locus is denoted by

$$i_n := \bigoplus_{\lambda \vdash n} i_\lambda : (\mathrm{Hilb}^n(\mathbb{C}^2))^T \hookrightarrow \mathrm{Hilb}^n(\mathbb{C}^2).$$

By the localization theorem in equivariant cohomology, the direct image

$$i_{n!} : H_T^*((\mathrm{Hilb}^n(\mathbb{C}^2))^T)' \to H_T^*(\mathrm{Hilb}^n(\mathbb{C}^2))'$$

is an isomorphism. The inverse is given by

$$\alpha \mapsto \sum_{\lambda \vdash n} \frac{i_\lambda^* \alpha}{c_{max}^T(T_{\xi_\lambda} \mathrm{Hilb}^n(\mathbb{C}^2))} 1_{\xi_\lambda}.$$

Let θ be the representation of T of weight 1. The isomorphism $H_T^*(pt) \cong \mathbb{Q}[u]$ is given by the first Chern class so for $a \in \mathbb{Z}$ we have

$$c_{tot}^T(\theta^{\otimes a}) = 1 + auZ \in H^*(B_T)[Z],$$

where $c_{tot}^T := 1 + c_1^T Z + c_2^T Z^2 + \cdots$ is the total Chern class. Then, by the properties of the Chern classes we get:

$$c_{tot}^T(i_\lambda^* F) = \prod_{i=1}^r \left(1 + f_i^\lambda uZ\right).$$

In particular, we know (see Nakajima [39]) that the representation of the fibre at ξ_λ of the tangent space of $\mathrm{Hilb}^n(\mathbb{C}^2)$ is given by

$$T_{\xi_\lambda} \mathrm{Hilb}^n(\mathbb{C}^2) \cong \bigoplus_{x \in D(\lambda)} \left(\theta^{h(x)} \oplus \theta^{-h(x)}\right).$$

It follows that $c_{max}^T(T_{\xi_\lambda} \mathrm{Hilb}^n(\mathbb{C}^2)) = (-1)^n h(\lambda)^2 u^{2n}$. The inverse localization formula gives then in $H_T^*(\mathrm{Hilb}^n(\mathbb{C}^2))'[Z]$:

$$c_{tot}^T(F) = (-1)^n \frac{1}{u^{2n}} \sum_{\lambda \vdash n} \frac{1}{h(\lambda)^2} \prod_{i=1}^r \left(1 + f_i^\lambda uZ\right) [\xi_\lambda]_T.$$

Since $H^q(\mathrm{Hilb}^n(\mathbb{C}^2)) = 0$ for $q > 2n$, we suppose $k \leq n$. The Z^k-term gives in $H_T^*(\mathrm{Hilb}^n(\mathbb{C}^2))'$:

$$u^{2n} c_k^T(F) = (-1)^n u^k \sum_{\lambda \vdash n} \frac{1}{h(\lambda)^2} \sigma_k(f_1^\lambda, \ldots, f_r^\lambda)[\xi_\lambda]_T,$$

where the $\sigma_k(-)$ are the elementary symmetric functions. Since u is invertible in the localized module, we get in $H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))'$:

$$u^{n-k} c_k^T(F) = \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \sigma_k\left(f_1^\lambda, \ldots, f_r^\lambda\right) [\lambda].$$

Since multiplication by u is an isomorphism after $H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))$, this equation is in fact an equation in $H_T^{2n}(\mathrm{Hilb}^n(\mathbb{C}^2))$.

Lemma 4.3. $J\left(\mathrm{gr}_k\left(u^{n-k} c_k^T(F)\right)\right) = c_k(F)$.

Proof of the lemma. With the Leray-Hirsch decomposition:

$$H_T^{2k}(\mathrm{Hilb}^n(\mathbb{C}^2)) \cong \bigoplus_{j=0}^k H^{2j}(B_T) \otimes H^{2k-2j}(\mathrm{Hilb}^n(\mathbb{C}^2))$$

we can write $c_k^T(F) = \sum_{j=0}^{k} u^j \otimes \alpha_j$ and $u^{n-k} c_k^T(F) = \sum_{j=0}^{k} u^{n-k+j} \otimes \alpha_j$. Then $\mathrm{gr}_k\left(u^{n-k}c_k^T(F)\right) = u^{n-k} \otimes \alpha_0$ and $J\left(\mathrm{gr}_k\left(u^{n-k}c_k^T(F)\right)\right) = \alpha_0$. Since $j^*u = 0$, we also have $c_k(F) = j^* c_k^T(F) = \alpha_0$. □

From this lemma, the commutativity of the diagram and the proposition 4.1 we obtain the expression of the Chern classes of F in Λ^n:

$$c_k(F) = \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \sigma_k(f_1^\lambda, \dots, f_r^\lambda)[s_\lambda]_k,$$

where $[s_\lambda]_k$ means that we keep only the component of cohomological degree k. The inverse Frobenius formula gives then:

$$c_k(F) = \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \sigma_k(f_1^\lambda, \dots, f_r^\lambda) \sum_{\substack{\mu \vdash n \\ l(\mu) = n-k}} z_\mu^{-1} \chi_\mu^\lambda p_\mu.$$

Similarly, starting from the formula (see also [33])

$$ch_k^T(i_\lambda^* F) = \frac{1}{k!} \sum_{i=1}^{r} (f_i^\lambda)^k,$$

we find

$$u^{n-k} ch_k^T(F) = \frac{1}{k!} \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \sum_{i=1}^{r} \left(f_i^\lambda\right)^k [\lambda]$$

and the naturality $j^* ch_k^T(F) = ch_k(F)$ implies in a similar manner:

$$ch_k(F) = \frac{1}{k!} \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \sum_{i=1}^{r} \left(f_i^\lambda\right)^k [s_\lambda]_k.$$

□

Remark 4.4. *The same method gives the Todd classes and with these formulas we can recover the well-known formula* [32, Proposition 5.2] *for the Chern classes of the tautological bundle B_n over* $\mathrm{Hilb}^n(\mathbb{C}^2)$ *(see* [3]*).*

Let t be an indeterminate, define $\omega_t p_k = t^{k-1} p_k$ and extend the definition to an algebra homomorphism $\omega_t : \Lambda \to \Lambda[t]$. Then $\omega_t p_\lambda = t^{|\lambda| - l(\lambda)} p_\lambda$: this notation takes care of the cohomological degree. In particular,

$$\omega_t s_\lambda = \sum_{k \geq 0} [s_\lambda]_k t^k.$$

Corollary 4.5. *The total Chern character of a T-linearized vector bundle F on* $\mathrm{Hilb}^n(\mathbb{C}^2)$ *is*

$$ch(F) = \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \operatorname{Coeff}\left(t^0, \omega_t s_\lambda F_\lambda(e^{1/t})\right).$$

The total Chern character of the dual bundle F^* is

$$ch(F^*) = \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \operatorname{Coeff}\left(t^0, \omega_t s_\lambda F_\lambda(e^{-1/t})\right).$$

Proof. By theorem 4.2 and his proof, the total Chern character of F is:

$$ch(F) = \sum_{\lambda \vdash n} \frac{1}{h(\lambda)} \sum_{i=1}^{r} \sum_{k \geq 0} \frac{1}{k!} \left(f_i^\lambda\right)^k [s_\lambda]_k.$$

By $F_\lambda(e^{1/t}) = \sum_{i=1}^{r} e^{f_i^\lambda/t} = \sum_{i=1}^{r}\sum_{k\geq 0}\frac{1}{k!}\left(f_i^\lambda\right)^k t^{-k}$ and $\omega_t s_\lambda = \sum_{k\geq 0}[s_\lambda]_k t^k$ we deduce the first formula. The second formula is similar since $ch_k(F^*) = (-1)^k ch_k(F)$. □

5. Comparison problems

5.1. **Poincaré duality problem.**

Denote by $\vartheta_{\overline{X_\mu}} \in H_*^{\mathrm{BM}}(\mathrm{Hilb}^n(\mathbb{C}^2))$ the homology fundamental class of the closed subvariety $\overline{X_\mu}$. By definition, $D\left[\overline{X_\mu}\right] = \vartheta_{\overline{X_\mu}}$ and these classes form the natural basis in homology as in Kaledin [31]. We get the *dual McKay correspondence* by defining a bijection $H_*^{\mathrm{BM}}(\mathrm{Hilb}^n(\mathbb{C}^2)) \to \mathcal{C}(S_n)$ with $\left[\overline{X_\mu}\right] \mapsto \chi_\mu$. Composing with the Frobenius morphism and introducing a sign (for a reason that will appear later) we define:

$$\begin{array}{rccc} \phi: & \Lambda^n & \to & H_*^{\mathrm{BM}}(\mathrm{Hilb}^n(\mathbb{C}^2)) \\ & p_\mu & \mapsto & (-1)^{n-l(\mu)} z_\mu \left[\overline{X_\mu}\right] \end{array}$$

With these notations, the " Poincaré duality problem " consists in the computation of the dotted arrow γ:

$$\begin{tikzcd} \Lambda^n \arrow[r, "\phi"] & H_*^{\mathrm{BM}}(\mathrm{Hilb}^n(\mathbb{C}^2)) \\ \Lambda^n \arrow[u, dotted, "\gamma"] \arrow[r, "\Psi"] & H^*(\mathrm{Hilb}^n(\mathbb{C}^2)) \arrow[u, "D"'] \end{tikzcd}$$

Proposition 5.1. *For $\mu \vdash n$,*

$$\gamma(p_\mu) = \frac{(-1)^{n-l(\mu)}}{\prod_{i\geq 1}\mu_i} p_\mu .$$

Proof. Let $\mu = (\mu_1, \ldots, \mu_k) = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$ be a partition of n. By definition, $\Psi(p_\mu) = q_\mu$ and with formula (2) we have $\Psi(p_\mu) = \left(\prod_{i\geq 1}\alpha_i!\right)\left[\overline{X_\mu}\right]$. Since we have $z_\mu = \left(\prod_{i\geq 1}\alpha_i!\right)\left(\prod_{i\geq 1}\mu_i\right)$, we get the result. □

5.2. **Chern character problem.**

With our notations, the " Chern character problem " consists in the computation of the dotted arrow Γ:

$$\begin{tikzcd} \Lambda^n \arrow[r, "\Psi"] & H^*(\mathrm{Hilb}^n(\mathbb{C}^2)) \\ \Lambda^n \arrow[u, dotted, "\Gamma"] \arrow[r, "\Theta"] & K(\mathrm{Hilb}^n(\mathbb{C}^2)) \arrow[u, "ch"'] \end{tikzcd}$$

Theorem 5.2. *For $\mu \vdash n$,*

$$\Gamma(p_\mu) = \frac{(-1)^{n-l(\mu)}}{\prod_{i\geq 1}\mu_i} p_\mu + \sum_{\substack{\nu\vdash n \\ l(\nu)<l(\mu)}} g_{\mu,\nu} p_\nu$$

for some coefficients $g_{\mu,\nu}$.

Proof. By the proposition 3.1, the map Γ is characterized by:

$$\Gamma(s_\mu) = ch(P_\mu^*).$$

We use the proposition 3.2 and we can apply corollary 4.5 with $F = P_\mu$ and $F_\lambda(s) = \widetilde{K}_{\mu,\lambda}\Big|_{t=s,q=s^{-1}}$:

$$\Gamma(s_\mu) = ch(P_\mu^*) = \sum_{\lambda\vdash n} \frac{1}{h(\lambda)} \operatorname{Coeff}\left(t^0, \omega_t s_\lambda \widetilde{K}_{\mu,\lambda}(e^{1/t}, e^{-1/t})\right).$$

Since $\widetilde{K}_{\mu,\lambda} = \langle \widetilde{H}_\lambda, s_\mu\rangle$ we get:

$$\Gamma(s_\mu) = \sum_{\lambda\vdash n} \frac{1}{h(\lambda)} \operatorname{Coeff}\left(t^0, \omega_t s_\lambda \langle \widetilde{H}_\lambda(e^{1/t}, e^{-1/t}), s_\mu\rangle\right)$$

and by linearity:

$$\Gamma(p_\mu) = \sum_{\lambda\vdash n} \frac{1}{h(\lambda)} \operatorname{Coeff}\left(t^0, \omega_t s_\lambda \langle \widetilde{H}_\lambda(e^{1/t}, e^{-1/t}), p_\mu\rangle\right).$$

We use formula (1):

$$\widetilde{H}_\lambda(q, q^{-1}) = q^{-n(\lambda)} \prod_{x\in D(\lambda)} \left(1 - q^{h(x)}\right) s_\lambda\left[\frac{X}{1-q}\right],$$

and find:

$$\Gamma(p_\mu) = \sum_{\lambda\vdash n} \tfrac{1}{h(\lambda)} \operatorname{Coeff}\left(t^0,\, (\omega_t s_\lambda) q^{-n(\lambda)} \prod_{x\in D(\lambda)} \left(1 - q^{h(x)}\right) \left\langle s_\lambda\left[\tfrac{X}{1-q}\right], p_\mu\right\rangle\Big|_{q=e^{1/t}}\right).$$

Observe that:

$$\left\langle s_\lambda\left[\frac{X}{1-q}\right], p_\mu\right\rangle = \left\langle s_\lambda, p_\mu\left[\frac{X}{1-q}\right]\right\rangle = \prod_{i=1}^{l(\mu)} \frac{1}{(1-q^{\mu_i})}\chi_\mu^\lambda,$$

so that in fact:

$$\Gamma(p_\mu) = \sum_{\lambda\vdash n} \frac{1}{h(\lambda)} \operatorname{Coeff}\left(t^0, (\omega_t s_\lambda)\chi_\mu^\lambda \left(q^{-n(\lambda)} \frac{\prod_{x\in D(\lambda)} (1 - q^{h(x)})}{\prod_{i=1}^{l(\mu)} (1 - q^{\mu_i})}\right)\Bigg|_{q=e^{1/t}}\right).$$

Since by construction:

$$\omega_t s_\lambda = \sum_{\nu\vdash n} z_\nu^{-1} \chi_\nu^\lambda p_\nu t^{n-l(\nu)},$$

the decomposition of $\Gamma(p_\mu)$ in the basis $\{p_\nu\}$ is (with $u = 1/t$):

$$\Gamma(p_\mu) = \sum_{\nu\vdash n} z_\nu^{-1} \sum_{\lambda\vdash n} \tfrac{1}{h(\lambda)} \chi_\nu^\lambda \chi_\mu^\lambda \operatorname{Coeff}\left(u^{n-l(\nu)}, \left(q^{-n(\lambda)} \frac{\prod_{x\in D(\lambda)} (1-q^{h(x)})}{\prod_{i=1}^{l(\mu)} (1-q^{\mu_i})}\right)\Bigg|_{q=e^u}\right) p_\nu.$$

Lemma 5.3.

$$\left(q^{-n(\lambda)} \frac{\prod_{x\in D(\lambda)} (1 - q^{h(x)})}{\prod_{i=1}^{l(\mu)} (1 - q^{\mu_i})}\right)\Bigg|_{q=e^u} = \frac{(-1)^{n-l(\mu)} h(\lambda)}{\prod_{i=1}^{l(\mu)} \mu_i} u^{n-l(\mu)} + \text{ upper powers }.$$

Proof of the lemma. By Taylor expansion we have:

$$1-e^{\mu_i u}=-\mu_i u(1+u(\ldots)),$$
$$1-e^{h(x)u}=-h(x)u(1+u(\ldots)),$$

so the first term in the Taylor expansion of the expression is:

$$\frac{\prod\limits_{x\in D(\lambda)}(-h(x)u)}{\prod\limits_{i=1}^{l(\mu)}(-\mu_i u)}=\frac{(-1)^{n-l(\mu)}h(\lambda)}{\prod\limits_{i=1}^{l(\mu)}\mu_i}u^{n-l(\mu)}.$$

□

We deduce that if $l(\nu)>l(\mu)$ then the coefficient of $\Gamma(p_\mu)$ at p_ν is zero. If $l(\nu)=l(\mu)$ then the coefficient of $\Gamma(p_\mu)$ at p_ν is:

$$g_{\mu,\nu}:=\frac{(-1)^{n-l(\mu)}}{\prod\limits_{i=1}^{l(\mu)}\mu_i}z_\nu^{-1}\sum_{\lambda\vdash n}\chi_\nu^\lambda\chi_\mu^\lambda.$$

In the Frobenius formula $p_\mu=\sum\limits_{\lambda\vdash n}\chi_\mu^\lambda s_\lambda$, we take the scalar product with p_ν to find the identity

$$\delta_{\mu,\nu}z_\nu=\sum_{\lambda\vdash n}\chi_\nu^\lambda\chi_\mu^\lambda,$$

which shows that $g_{\mu,\nu}=0$ if $\mu\neq\nu$ and $g_{\mu,\mu}=\dfrac{(-1)^{n-l(\mu)}}{\prod\limits_{i=1}^{l(\mu)}\mu_i}$. □

5.3. **Conclusion.**

• The space $K(\mathrm{Hilb}^n(\mathbb{C}^2))$ has a decreasing *topological* filtration defined by the codimension of the support of coherent sheaves:

$$F_dK(\mathrm{Hilb}^n(\mathbb{C}^2)):=\mathbb{Q}\left\{\mathcal{F}\mid \operatorname{codim}\operatorname{Supp}\mathcal{F}\geq d\right\}.$$

The Chern character $ch:K(\mathrm{Hilb}^n(\mathbb{C}^2))\to H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$ is compatible with this filtration and the induced graded map $\operatorname{gr}ch:\operatorname{gr}K(\mathrm{Hilb}^n(\mathbb{C}^2))\to H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$ is given by the cohomology class of the support (see [11, §5.9]):

$$\operatorname{gr}ch(\mathcal{F})=[\operatorname{Supp}\mathcal{F}].$$

Denote the decreasing cohomological filtration on Λ by:

$$F_d\Lambda^n:=\mathbb{Q}\left\{p_\lambda\mid \deg p_\lambda\geq d\right\},$$

and $\operatorname{gr}_F\Lambda^n$ the induced graded vector space. Then our preceding results mean:

Theorem 5.4. *The McKay correspondence Θ is compatible with the topological filtration of $K(\mathrm{Hilb}^n(\mathbb{C}^2))$ and the decreasing filtration of Λ^n.*

Proof. By formula (2) the cohomology classes $\left[\overline{X_\mu}\right]$ form a homogeneous basis of $H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$, so the classes of the structural sheaves $\mathcal{O}_{\overline{X_\mu}}$ form a graded basis of $\operatorname{gr}K(\mathrm{Hilb}^n(\mathbb{C}^2))$ with $\deg\mathcal{O}_{\overline{X_\mu}}=n-l(\mu)$. Then the theorem 5.2 implies that $\Theta^{-1}\left(\mathcal{O}_{\overline{X_\mu}}\right)\in F_{n-l(\mu)}\Lambda^n$ and after inversion of the matrix we get the result. □

• By theorem 5.2, the induced graded map:

$$\operatorname{gr}\Gamma=\Psi\circ\operatorname{gr}ch\circ\operatorname{gr}\Theta:\operatorname{gr}_F\Lambda^n\to\operatorname{gr}K(\mathrm{Hilb}^n(\mathbb{C}^2))\to H^*(\mathrm{Hilb}^n(\mathbb{C}^2))\to\operatorname{gr}^F\Lambda^n$$

is defined by $\operatorname{gr}\Gamma(p_\mu) = \dfrac{(-1)^{n-l(\mu)}}{\prod_{i\geq 1}\mu_i} p_\mu$, which is exactly the same formula as for the map γ (this justifies our sign modification in §5.1). The number $\operatorname{age}(\lambda) := n - l(\lambda)$ is the *age* (or *shifting degree*) of any $\sigma \in S_n$ of cycle-type λ for the permutation action on $\mathbb{C}^{2n}$, defined as follows: the eigenvalues of σ on $\mathbb{C}^{2n}$ are squares of the unit $e^{2\mathbf{i}\pi r_j}$, $r_j \in [0,1[$ and by definition $\operatorname{age}(\sigma) := \sum_{j\geq 1} r_j = n - l(\lambda)$. The number $\langle\sigma\rangle := \prod_{i\geq 1}\lambda_i$ can be interpreted as the product of the dimensions of the orbit vector subspaces one gets by performing a Frobenius decomposition of the matrix of σ on $\mathbb{C}^{2n}$. We call this number $\langle\sigma\rangle$ the *complexity degree* of σ (see [3]). Then, the map $p_\lambda \mapsto \frac{(-1)^{\operatorname{age}(\lambda)}}{\langle\lambda\rangle} p_\lambda$ has a signification for any finite group G acting on a vector space V : denoting by $\chi_{[g]}$ the class function of the conjugacy class $[g]$ of $g \in G$, this map $\mathcal{C}(G) \to \mathcal{C}(G)$ is $\chi_{[g]} \mapsto \frac{(-1)^{\operatorname{age}(g)}}{\langle g\rangle}\chi_{[g]}$.

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